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Note

Recursion Relations for the Group $SU(2)$

INTRODUCTION

The general expression for the matrix elements of the irreducible representations of group $SU(2)$ is [1]

$$D^{(j)}(a, b)_{k,m} = \left| \frac{(j+k)!\,(j-k)!}{(j+m)!\,(j-m)!} \right|^{1/2} \sum_{\nu} \binom{j+m}{\nu} \binom{j-m}{k-m+\nu} a^{j+m-\nu} a^{*j-k-\nu} b^{\nu} (-b^{*})^{k-m+\nu}, \tag{1}$$

where ν assumes every value that makes the exponents positive or null and $|a|^2 + |b|^2 = 1$.

Algebraic and numerical calculations of the $D^{(j)}(a, b)_{k,m}$ matrix elements through Eq. (1) involve large numbers of terms, even for relatively small values of j. It seems to us that, in order to simplify the computation of $D^{(j)}(a, b)$ matrices, it would be very useful to have available some simple recursion relations. Since $SU(2)$ is the universal covering group of $SO(3)$ [2], recursion relations for $SU(2)$ can be easily transformed for use with $SO(3)$, by expressing the parameters a and b in a convenient form. For example, using Euler angles [3] the parameters a and b are given by

$$\begin{aligned} a &= \cos \beta/2 \exp\{-i(\alpha + \gamma)/2\}, \\ b &= \sin \beta/2 \exp\{i(\alpha - \gamma)/2\}. \end{aligned} \tag{2}$$

If we consider the parametrization of the group $SO(3)$ in terms of an angle μ measured counterclockwise around a unitary direction $\boldsymbol{\lambda}$ [4–5] we obtain

$$\begin{aligned} a &= \cos \mu/2 - i\lambda_z \sin \mu/2, \\ b &= (\lambda_y - i\lambda_x) \sin \mu/2. \end{aligned} \tag{3}$$

Expressions for the parameters a and b using other parametrizations for the group $SO(3)$ are always possible to obtain, matching the $D(1/2)$ corresponding representations for both groups.

In this paper, we show that recursion relations can be obtained for any group,

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from the Clebsch–Gordan decomposition of its Kronecker product. In particular, for the simple reducible $SU(2)$ group [6] the Kronecker product results:

$$D^{(j_1)}_{k_3-k_2,m_3-m_2} D^{(j_2)}_{k_2,m_2} = \sum_{j_3=|j_1-j_2|}^{j_1+j_2} \langle j_1k_3 - k_2\,, j_2k_2 \mid j_3k_3\rangle\, D^{(j_3)}_{k_3,m_3} \langle j_1m_3 - m_2\,, j_2m_2 \mid j_3m_3\rangle \tag{4}$$

or

$$\langle j_1k_3 - k_2\,, j_2k_2 \mid j_3k_3\rangle\, D^{(j_3)}_{k_3,m_3} = \sum_{m_2=-j_2}^{j_2} \langle j_1m_3 - m_2\,, j_2m_2 \mid j_3m_3\rangle\, D^{(j_1)}_{k_3-k_2,m_3-m_2} D^{(j_2)}_{k_3-k_2,m_3-m_2}, \tag{5}$$

where

$$D^{(j)}_{k,m} = D^{(j)}(a, b)_{k,m}\,, \tag{6}$$

Equation (5) was obtained from Eq. (4) using the orthogonality relations of the Clebsch–Gordan coefficients.

The simplest nontrivial expressions for Eqs. (4) and (5) are obtained in the next section for the cases $j_2 = 1$ and $j_2 = \frac{1}{2}$. The analytic expressions of the Clebsch–Gordan coefficients for these values were taken from Condon and Shortley [7].

Recursion Relations

(a) Case $j_1 = j, j_2 = 1, j_3 = j + 1$.

Among the values of k_2 and m_2 we select $k_2 = m_2 = 0$. Then, $k_1 = k_3 = k$ and $m_1 = m_2 = m$. Equation (2) gives for ranging j

$$\begin{aligned} j[[(j+1)^2 - k^2][(j+1)^2 - m^2]]^{1/2} D^{(j+1)}_{k,m} &= (2j+1)[j(j+1)(|a|^2 - |b|^2) - km]^{1/2} D^{(j)}_{k,m} \\ &\quad - (j+1)[(j^2-k^2)(j^2-m^2)]^{1/2} D^{(j-1)}_{k,m}. \end{aligned} \tag{7}$$

The use of Eq. (7) in computational calculations has been discussed by Walker [8] for the particular case $a = \cos(\beta/2)$ and $b = \sin(\beta/2)$.

(b) Case $j_1 = j, j_2 = 1, j_3 = j - 1$.

This case is similar to case (a) with a simple redefinition where $j_1 = j - 1$ and $j_3 = j$.

(c) Case $j_1 = j, j_2 = 1, j_3 = j$.

For fixed j and ranging k and m, Eq. (5) gives:

$$\mathbf{V}' = \mathbf{M} \cdot \mathbf{V}, \tag{8}$$

where

$$\mathbf{V}' = \begin{pmatrix} -C_+(k)\, D^{(j)}_{k+1,m} \\ kD^{(j)}_{k,m} \\ C_-(k)\, D^{(j)}_{k-1,m} \end{pmatrix}, \tag{9}$$

$$\mathbf{M} = \begin{pmatrix} a^2 & -2ab^* & b^{*2} \\ ab & |a|^2 - |b|^2 & -a^*b^* \\ b^2 & 2a^*b & a^{*2} \end{pmatrix}, \tag{10}$$

$$\mathbf{V} = \begin{pmatrix} -C_-(m)\, D^{(j)}_{k,m-1} \\ mD^{(j)}_{k,m} \\ C_+(m)\, D^{(j)}_{k,m+1} \end{pmatrix}, \tag{11}$$

and

$$C_\pm(k) = [j(j+1) - k(k \pm 1)]^{1/2}. \tag{12}$$

Equation (8) is not convenient for numerical calculations because it fails for $|a| = 0$ and $|b| = 0$. Nevertheless, it is possible, from this equation and its inverse, to obtain relations where the cross products ab have been eliminated, namely:

$$D^{(j)}_{j,k} = [-C_+(k)\, b^* D^{(j)}_{j,k+1}]/[(j-k)a], \tag{13}$$

$$D^{(j)}_{k-1,m} = [(k+m)\, bD^{(j)}_{k,m} + C_+(m)\, a^* D^{(j)}_{k,m+1}]/[C_-(k)a], \tag{14}$$

$$D^{(j)}_{k,-j} = [-C_+(k)\, a^* D^{(j)}_{k+1,-j}]/[(j-k)\, b^*], \tag{15}$$

$$D^{(j)}_{m,-k+1} = -[(k+m)\, aD^{(j)}_{m,-k} + C_+(m)\, bD^{(j)}_{m+1,-k}]/[C_-(k)\, b^*]. \tag{16}$$

Equations (13) to (16) are similar to those reported by Fano and Racah [9] when we take $a = \cos\theta/2$ and $b = -\sin\theta/2$.

(d) Case $j_1 = j - 1/2, j_2 = 1/2, j_3 = j$.

For ranging j, using Eq. (5) we obtain

$$D^{(j)}_{k,m} = \left(\frac{j+m}{j+k}\right)^{1/2} aD^{(j-1/2)}_{k-1/2,m-1/2} - \left(\frac{j-m}{j+k}\right)^{1/2} b^* D^{(j-1/2)}_{k-1/2,m+1/2}\,. \tag{17}$$

Calculations can be reduced if we use the well-known symmetry property:

$$D^{(j)}(a,b)_{-k,-m} = (-)^{k-m}\, D^{(j)*}(a,b)_{k,m}\,. \tag{18}$$

Two other useful relations can be deduced from Eq. (1), namely,

$$D^{(j)}(a, b)_{-m,-k} = D^{(j)}(a^*, b)_{k,m}$$
$$= (a^*/a)^{k+m} D^{(j)}(a, b)_{k,m}, \tag{19}$$
$$D^{(j)}(a, b)_{m,k} = D^{(j)}(a, -b^*)_{k,m}$$
$$= (-b/b^*)^{k-m} D^{(j)}(a, b)_{k,m}. \tag{20}$$

The symmetry properties given by Eqs. (18) to (20) reduce the number of unknown matrix elements of $D^{(j)}(a, b)$ matrices to $j(j+2)$ for integral j and $j(j+2) - 1/4$ for half-integral j.

Discussion

The Clebsch–Gordan decomposition of the Kronecker product of the group $SU(2)$ is a source of recursion relations, and in this paper we have shown only those of lowest order. Evidently, the method can be applied to other more sophisticated groups although in general those groups are not simply reducible and therefore recursion relations will be dependent on the particular choice of the Clebsch–Gordan coefficients.

In order to bound the numerical errors of the complex recursion relations we have constructed a subroutine DMATRX [10] in FORTRAN IV language using Eqs. (13) and (14) where $|a| \geqslant |b|$, (15) and (16) in the other case, and (18) to (20) in order to reduce the number of iterations.

The errors obtained checking the unitarity of the matrices and the closure property of the group are very similar and they have a maximum when $|a| = |b|$. In this case, and for $j < 16$, we lose at most four significant figures using one of the three precisions of the IBM 370; namely, single, double, and extended.

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